The $O(n)$ Loop Model on the 3–12 Lattice

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The partition function of the $O(n)$ loop model on the honeycomb lattice is mapped to that of the $O(n)$ loop model on the 3–12 lattice. Both models share the same operator content and thus critical exponents. The critical points are related via a simple transformation of variables. When $n = 0$ this gives the recently found exact value $\mu = 1.711041...$ for the connective constant of self-avoiding walks on the 3–12 lattice. The exact critical points are recovered for the Ising model on the 3–12 lattice and the dual asanoha lattice at $n = 1$.

1. INTRODUCTION

Consider the $O(n)$ loop model on the honeycomb lattice with $\mathcal{N}_h$ vertices. The partition function is defined by

$$Z_h = \sum_{\text{loops}} t^{\mathcal{N}_h - L} n^P \tag{1}$$

where the sum is over all configurations of non-intersecting loops. In any given configuration L is the number of occupied vertices, with $t^{\mathcal{N}_h - L}$ the weight of the empty (unoccupied) vertices. The variable n is the fugacity of a closed loop, with P their number in any given configuration. A typical loop configuration is shown in Fig. 1.

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Fig. 1. Loop configuration on the honeycomb lattice with weight $t^5 n^2$.

This model originates from the high-temperature expansion[1] of the $O(n)$ or n-vector model.[2] Nienhuis identified two branches of critical points for the $O(n)$ loop model on the honeycomb lattice defined by[3]

$$(2 - t^2)^2 = 2 - n \tag{2}$$

Baxter has shown that this is also the necessary condition for which the row transfer matrix of the corresponding 3-state vertex model can be diagonalised by means of the co-ordinate Bethe Ansatz.[4] This vertex model turned out to be a special case of a more general solvable model defined on the square lattice.[5] The Bethe Ansatz solution was later extended to open boundary conditions.[6, 7]

Most importantly, the $O(n)$ loop model on the honeycomb lattice has led to a wealth of exact information on the configurational properties of selfavoiding walks in the $n \to 0$[8] limit. For a review, we refer the reader to ref. (9). The simplest result concerns the enumeration of a single N-step self-avoiding walk from a point deep in the bulk, for which the number of configurations scales as

$$C_N \sim \mu^N N^{\gamma - 1} \tag{3}$$

for large N. The connective constant, $\mu = \sqrt{2 + \sqrt{2}} = 1.847759...$, follows from (2). The configurational exponent $\gamma = 43/32$ was first obtained via Coulomb gas calculations.[3] More general configurational exponents have been obtained for arbitrary networks of long chains, both in the bulk and near a surface.[9] [2]

Given the solvability of the $O(n)$ loop model on the honeycomb lattice along the line of critical points defined by (2), I had often wondered if the corresponding model could be solved on the 3–12 lattice depicted in Fig. 2. It also has co-ordination number three. Prompted by a suggestion that the

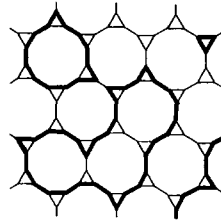[2] The most recent developments are reported in ref. (10).

Fig. 2.   Loop configuration on the 3-12 lattice.

connective constant for self-avoiding walks on the 3-12 lattice may also be obtained exactly, Jensen and Guttmann have found the value $\mu =$ 1.711041....[11] They were able to relate the generating functions for both self-avoiding walks and self-avoiding polygons on the honeycomb lattice to those on the 3-12 lattice by a simple change of variables. This mapping is discussed here in the context of the more general $O(n)$ loop model, which indeed turns out to be solvable at criticality.

2. *O*(*n*) LOOP MODEL ON THE 3-12 LATTICE

Vertices on the honeycomb lattice are either empty or occupied. For each type of vertex configuration on the honeycomb lattice there are two possible configurations on the 3-12 lattice, as shown in Fig. 3. It thus follows that any given configuration of loops on the honeycomb lattice, with weight $t^{-i_h - L} n^P$, maps to $(t^3 + n_A)^{-i_h - L} (t + 1)^L n^P$ possible configurations on the 3-12 lattice, with $n_A = n$. One of the $(t^3 + n_A)^5 (t + 1)^{23} n^2$ possible configurations generated from the honeycomb configuration in Fig. 1 is shown in Fig. 2.

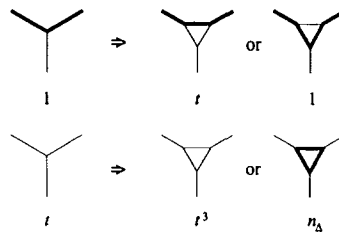The partition function of the $O(n)$ loop model on the 3-12 lattice can be written

Fig. 3.   Mapping between vertex configurations. The other possible vertex weights are similarly defined under uniform rotation.

$$Z_{3-12} = \sum_{\text{loops}} (t^3 + n_A)^{\mathcal{N}_h - L} (t+1)^L n^P$$

$$= (t^3 + n_A)^{\mathcal{N}_h} \sum_{\text{loops}} \left(\frac{t+1}{t^3 + n_A} \right)^L n^P \tag{4}$$

This is to be compared with

$$Z_h = t^{\mathcal{N}_h} \sum_{\text{loops}} \left(\frac{1}{t} \right)^L n^P \tag{5}$$

It follows that the critical points of the $O(n)$ loop model on the 3–12 lattice are given by solving (2) with

$$\frac{1}{t} \to \frac{t+1}{t^3 + n} \tag{6}$$

2.1. Self-Avoiding Walks

In particular, when $n = 0$

$$2 + 8t + 12t^2 + 8t^3 + 2t^4 - 4t^6 - 8t^7 - 4t^8 + t^{12} = 0 \tag{7}$$

The exact connective constant $\mu = 1.711041...$ follows from the largest real root, as obtained by Jensen and Guttmann.[11] Equivalently, it follows on solving

$$\frac{1}{\sqrt{2 + \sqrt{2}}} = \frac{1}{t^2} + \frac{1}{t^3} \tag{8}$$

Their mapping $x \to x^2 + x^3$ between generating functions follows on defining $x = 1/t$ as the weight per step.

2.2. The Ising Model

Another point of note is the Ising model at $n = 1$. It is known that the critical temperature of the Ising model on both the honeycomb and tri-angular lattices follows from (2) and the standard duality relation. In a similar way, the critical Ising point can be obtained from (6) for both the 3–12 lattice and its dual (the asanoha lattice). For $n = 1$ (2) and (6) give

$$\frac{1}{\sqrt{3}} = \frac{1}{t^2 - t + 1} \tag{9}$$

Taking the positive root gives

$$e^{2A_c} = \tfrac{1}{2}(1 + \sqrt{4\sqrt{3} - 3}) = 1.490984... \tag{10}$$

as the critical coupling on the dual asanoha lattice. The critical point

$$e^{2K_c} = \tfrac{1}{2}(3 + \sqrt{3} + \sqrt{12 + 10\sqrt{3}}) = 5.073446... \tag{11}$$

on the 3–12 lattice follows from the duality relation $e^{-2A_c} = \tanh K_c$. The Ising values (10) and (11) are precisely those given by Syozi,[12] who arrived at the Ising model on the 3–12 lattice from the Ising model on the honeycomb lattice after the successive application of the double decoration process and the star-triangle transformation.

3. CONCLUSION

The partition function of the $O(n)$ loop model on the honeycomb lattice has been mapped to that of the $O(n)$ loop model on the 3–12 lattice. The critical behaviour of both models is thus related. In particular, they share the same operator content and thus critical exponents. Although the mapping between the models is particularly simple, it nevertheless provides a clear example of universality between models defined on regular and semi-regular lattices. The non-universal features, such as the critical points, are related via the transformation (6). When $n = 0$ this gives the exact value $\mu = 1.711041...$ found recently by Jensen and Guttmann[11] for the connective constant of self-avoiding walks on the 3–12 lattice.[3] When $n = 1$ the exact critical points, (11) and (10) are recovered for the Ising model on the 3–12 lattice and the dual asanoha lattice.

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[3] Other non-universal quantities for self-avoiding walks on the 3-12 lattice can also be obtained from their honeycomb counterparts, such as the critical adsorption temperature at a boundary.[13]

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